Phase instabilities of oscillatory standing squares and alternating rolls

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The stability of standing squares and alternating rolls is investigated within the framework of the lowest-order amplitude equations describing the interaction of right-, left-, up-, and down-traveling waves on a square lattice. It is found that a standing-square or alternating-roll pattern is subject to a stationary rectangular phase instability and an oscillatory phase instability. The rectangular mode is locally equivalent to a stretching along one coordinate axis and contraction along the orthogonal axis. The oscillatory instability is locally equivalent to the coordinate axes rotating towards or away from each other, and leads to quasiperiodic temporal oscillations of the bifurcated state.

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I. INTRODUCTION

Standing squares and alternating rolls are oscillatory patterns with square symmetry. Both patterns are made up of a pair of two-dimensional standing waves at right angles to each other; for standing squares the two waves oscillate in phase, and for alternating rolls they are $\pi/2$ out of phase.

An oscillatory standing-rectangular planform has been observed in electrohydrodynamic convection of nematic liquid crystals; see, for example, Kai and Hirakawa [1], Joets and Ribotta [2] (for a similar pattern), and Nasuno, Sano, and Sawada [3]. Alternating rolls have been found to occur in three-dimensional magnetoconvection (Matthews [4], Matthews, Proctor, and Weiss [5], and Clune and Knobloch [6]). They are also found in binary convection; the transition from squares to rolls takes place via an alternating-roll solution. This has been observed by Moses and Steinberg [7,8] and Le Gal, Pocheau, and Croquette [9]. There has also been theoretical work on this system by Müller and Lücke [10], Pismen [11], and Armbruster [12].

Silber and Knobloch [13] have studied the problem of a Hopf bifurcation on a square lattice in great detail in the nonmodulational case. This has recently been extended to consider an anisotropic environment [14].

Coullet, Fauve, and Tirapegui [15] considered the phase instabilities of a two-dimensional standing-wave pattern, and found a large-scale oscillatory instability leading to a quasiperiodic regime. Pismen [16] studied the modulational instabilities of a two-dimensional standing wave, and also found a general form for the stability criteria of other standing-wave patterns. The phase instabilities of standing squares and alternating rolls have not yet been investigated, although Nasuno, Sano, and Sawada [3] observed structures, which they termed phase waves, in their experiments on electrohydrodynamic convection of nematic liquid crystals. These provide beautiful examples of a planform modulated on a long length scale, and the current work was conceived as a first step in trying to understand their structure.

Here we use phase dynamics (Pomeau and Manneville [17] and Kuramoto [18]) to investigate the stability of standing squares and alternating rolls to long-wavelength modulational disturbances.

II. THE AMPLITUDE EQUATIONS

We can consider an oscillatory system with square symmetry to consist of four modulated waves, one traveling in each of the $\pm x$ and $\pm y$ directions. We assume that the system can be represented by a typical physical variable $f(x,y,t)$ which may be written

$$f(x,y,t)=A(X,Y,T)e^{i(\omega_c t-k_c x)}+B(X,Y,T)e^{i(\omega_c t+k_c x)}+C(X,Y,T)e^{i(\omega_c t-k_c y)}+D(X,Y,T)e^{i(\omega_c t+k_c y)}+\text{c.c.}+\text{h.o.t.}\ , \tag{1}$$

where $\omega_c(k_c)$ and k_c are the critical frequency and wave number respectively, for the onset of instability, where X,Y,T are long modulation scales in the x,y,t directions, respectively, and where h.o.t. denotes higher-order terms.

The general interaction of two orthogonal pairs of traveling waves close to the onset of the pattern-forming instability can be modeled by the following equations:

$$A_T=rA-cA_X+(1+i\alpha)A_{XX}+i\alpha A_{YY}-\nu|A|^2A-\mu|B|^2A-\gamma(|C|^2+|D|^2)A-\eta\bar{B}CD\ , \tag{2}$$

$$B_T=rB+cB_X+(1+i\alpha)B_{XX}+i\alpha B_{YY}-\nu|B|^2B-\mu|A|^2B-\gamma(|C|^2+|D|^2)B-\eta\bar{A}CD\ , \tag{3}$$

$$C_T = rC - cC_Y + (1+i\alpha)C_{YY} + i\alpha C_{XX} - \nu|C|^2C - \mu|D|^2C - \gamma(|A|^2+|B|^2)C - \eta AB\bar{D} , \quad (4)$$

$$D_T = rD + cD_Y + (1+i\alpha)D_{YY} + i\alpha D_{XX} - \nu|D|^2D - \mu|C|^2D - \gamma(|A|^2+|B|^2)D - \eta AB\bar{C} . \quad (5)$$

where higher order terms have been neglected. Here $r>0$, c, and α are real constants, while ν, μ, γ, and η are complex constants. All are assumed to be $O(1)$; however, this means that in physical dimensional units the control parameter r and the phase velocity c must be very small, since we assume that the amplitudes A, B, C, D and $\partial_x, \partial_y = 0(\epsilon)$, where $|\epsilon| \ll 1$, in order to justify using amplitude equations. Phase velocities of size $O(1)$ in physical units correspond to the limit of large c within the framework of these amplitude equations. This scaling is in agreement with Coullet, Fauve, and Tirapegui [15], but not with Pismen [16], who chose c to be $O(1)$ in physical units. No higher-order spatial derivatives are included in the amplitude equations, since it turns out that the leading-order phase behavior can be described fully without them. In particular, the growth rate of zigzag-type phase disturbances is dominated by the contribution from the imaginary transverse diffusion, i.e., the term $i\alpha A_{YY}$ in Eq. (2) and the corresponding terms in the other three equations.

The linear parts of the amplitude equations are recovered, after appropriate rescalings, from the most general isotropic expansion for the growth rate, σ, of a Fourier mode of finite wavelength undergoing a bifurcation at critical wave number $|\mathbf{k}|=k_c$, and frequency $\omega_c(k_c)$, under the influence of an external parameter r:

$$\sigma = r - \xi_0^2(|\mathbf{k}|^2-k_c^2)^2 + O((|\mathbf{k}|^2-k_c^2)^3) + i[\omega_c + \omega_1(|\mathbf{k}|^2-k_c^2) + O((|\mathbf{k}|^2-k_c^2)^2)] , \quad (6)$$

where ξ_0 and ω_1 are real constants.

The amplitude equations should respect the symmetries of the system. They must be invariant under X translation, Y translation, and time T translation, and must also conform to the symmetries of the square lattice: X reflection, Y reflection, and rotation through $\pi/2$. These symmetries determine the possible nonlinear terms.

The basic maximally symmetric solutions of Eqs. (2), (3), (4), and (5) are (a) the trivial solution, $A=B=C=D=0$; (b) traveling rolls, e.g., $A\neq 0$, $B=C=D=0$; (c) standing rolls, e.g., $A=B\neq 0$, $C=D=0$; (d) traveling squares, e.g., $A=C\neq 0$, $B=D=0$; (e) standing squares, $A=B=C=D\neq 0$; (f) alternating rolls, e.g., $A=B=iC=iD\neq 0$; and (g) standing cross rolls, $A=B\neq 0, C=D\neq 0, |A|\neq|C|$.

These solutions and their stability to nonmodulational perturbations were determined by Silber and Knobloch [13].

III. LINEAR STABILITY OF STANDING SQUARES

In this paper we will investigate standing squares. The full analysis also applies to alternating rolls, by means of the transformation $C\to iC, D\to iD$ (equivalent to $\chi\to\chi+\pi/2, \phi\to\phi+\pi/2$ below) and $\eta\to-\eta$, which leaves the amplitude equations unaltered.

We search for a steady solution of the form

$$A = R_0 e^{i(\Omega T - qX)} , \quad B = R_0 e^{i(\Omega T + qX)} , \quad C = R_0 e^{i(\Omega T - qY)} , \quad D = R_0 e^{i(\Omega T + qY)} , \quad (7)$$

where R_0, Ω, and q are constants. This represents a perfect standing-square pattern at a frequency and wave number differing slightly from the critical. From Eqs. (2)–(5), we obtain

$$R_0^2 = \frac{r-q^2}{\nu_r+\mu_r+\eta_r+2\gamma_r} , \quad (8)$$

$$\Omega = cq - \alpha q^2 - (r-q^2)\frac{(\nu_i+\mu_i+\eta_i+2\gamma_i)}{(\nu_r+\mu_r+\eta_r+2\gamma_r)} . \quad (9)$$

We require $R_0^2 \geq 0$, so $r=q^2$ is the neutral stability curve, where the standing-square solution bifurcates from the trivial solution.

First we consider a perturbation of the form

$$A = R_0(1+R)e^{i(\Omega T - qX + \theta)} , \quad (10)$$

$$B = R_0(1+S)e^{i(\Omega T + qX + \psi)} , \quad (11)$$

$$C = R_0(1+U)e^{i(\Omega T - qY + \chi)} , \quad (12)$$

$$D = R_0(1+V)e^{i(\Omega T + qY + \phi)} , \quad (13)$$

where R, S, U, V, θ, ψ, χ, and ϕ are functions of the slow time scale T only. In this case, the amplitude modes R, S, U, and V, and the phase-locking mode $(\theta+\psi-\chi-\phi)$ have nonzero growth rate eigenvalues; they are stable when

$$\mu_r+\nu_r+\eta_r+2\gamma_r > 0 , \quad (14)$$

$$\mu_r+\nu_r-3\eta_r-2\gamma_r > 0 , \quad (15)$$

$$\eta_r(\nu_r+\mu_r-\eta_r-2\gamma_r)+\eta_i(\nu_i+\mu_i-\eta_i-2\gamma_i) < 0 , \quad (16)$$

$$\nu_r-\mu_r-\eta_r > 0 . \quad (17)$$

In order for the trivial solution to be unstable, we also require that $r>0$.

The remaining modes have zero growth rate eigenvalues in the spatially uniform regime. If we now allow modulations, the perturbations R, S, U, V, θ, ψ, χ, and ϕ may be functions of all the slow variables X, Y, and T. Looking only for long-wavelength effects so that $|\partial_X|, |\partial_Y| \ll 1$, we may immediately slave the amplitude and phase-locking modes to the remaining phase modes. This leads to the following coupled phase equations:

$$(\theta-\psi)_T = A(\theta+\psi+\chi+\phi)_X + M(\theta-\psi)_{XX} + K(\theta-\psi)_{YY} - N(\chi-\phi)_{XY} + W(\theta+\psi+\chi+\phi)_{XT} + \text{h.o.t.} , \quad (18)$$

$$(\chi-\phi)_T = A(\theta+\psi+\chi+\phi)_Y + M(\chi-\phi)_{YY} + K(\chi-\phi)_{XX} - N(\theta-\psi)_{XY} + W(\theta+\psi+\chi+\phi)_{YT} + \text{h.o.t.} , \quad (19)$$

$$(\theta+\psi+\chi+\phi)_T=G\nabla^2(\theta+\psi+\chi+\phi)+H\{(\theta-\psi)_X+(\chi-\phi)_Y\}+Z\{(\theta-\psi)_{XT}+(\chi-\phi)_{YT}\}+\text{h.o.t.}\ , \tag{20}$$

where

$$F=c-2\alpha q\ , \tag{21}$$

$$A=-\tfrac{1}{2}(F+2q\lambda_1)\ , \tag{22}$$

$$B=\frac{A(F+2q\lambda_3)}{4R_0^2(\eta_r+\lambda_3\eta_i)}\ , \tag{23}$$

$$C=1+\alpha\lambda_1\ , \tag{24}$$

$$D_\pm=\frac{1}{4R_0^2}(\lambda_1F-2q)\left\{\frac{(2q\eta_r-F\eta_i)}{(\eta_r+\lambda_3\eta_i)f_3}\pm\frac{2q}{f_2}\right\}\ , \tag{25}$$

$$G=\tfrac{1}{2}+\frac{q}{2R_0^2f_1}(F\lambda_2-2q)+\alpha\lambda_2\ , \tag{26}$$

$$H=-F-2q\lambda_2\ , \tag{27}$$

$$K=\alpha\lambda_1\ , \tag{28}$$

$$M=B+C+D_+\ , \tag{29}$$

$$N=B+D_-\ , \tag{30}$$

$$W=\frac{q\lambda_1}{2R_0^2f_1} \tag{31}$$

$$Z=\frac{q\lambda_2}{R_0^2f_2} \tag{32}$$

$$f_1=\nu_r-\mu_r-\eta_r\ , \tag{33}$$

$$f_2=\nu_r+\mu_r+\eta_r+2\gamma_r\ , \tag{34}$$

$$f_3=\nu_r+\mu_r-\eta_r-2\gamma_r\ , \tag{35}$$

$$\lambda_1=f_1^{-1}(\nu_i-\mu_i-\eta_i)\ , \tag{36}$$

$$\lambda_2=f_2^{-1}(\nu_i+\mu_i+\eta_i+2\gamma_i)\ , \tag{37}$$

$$\lambda_3=f_3^{-1}(\nu_i+\mu_i-\eta_i-2\gamma_i)\ . \tag{38}$$

We can rewrite inequalities (14)–(17) in terms of the quantities defined above, yielding

$$f_1>0\ , \tag{39}$$

$$f_2>0\ , \tag{40}$$

$$f_3(\eta_r+\lambda_3\eta_i)<0\ , \tag{41}$$

$$f_3-2\eta_r>0\ . \tag{42}$$

We are looking for instabilities associated with the two spatial phases $\phi^x=-(\theta-\psi)/2(k_c+q)$ and $\phi^y=-(\chi-\phi)/2(k_c+q)$, and the temporal phase $\phi^t=(\theta+\psi+\chi+\psi)/4(\omega_c+\Omega)$.

Writing $(\theta-\psi)=(\hat\theta-\hat\psi)e^{\sigma T+i(kX+lY)}$ + c.c., etc., we find that the growth rates σ of the three eigenmodes are given by

$$\sigma_1=-2(M+N-K)\frac{k^2l^2}{k^2+l^2}-K(k^2+l^2)+O(k^4)\ , \tag{43}$$

$$\sigma_2=i(AH)^{1/2}(k^2+l^2)^{1/2}-\tfrac{1}{2}(G+M+HW+AZ)(k^2+l^2)+(M+N-K)\frac{k^2l^2}{k^2+l^2}+\hat\sigma+O(k^4)\ , \tag{44}$$

$$\sigma_3=-i(AH)^{1/2}(k^2+l^2)^{1/2}-\tfrac{1}{2}(G+M+HW+AZ)(k^2+l^2)+(M+N-K)\frac{k^2l^2}{k^2+l^2}-\hat\sigma+O(k^4)\ , \tag{45}$$

where $\hat\sigma$ is an $O(k^3)$ term whose form cannot be determined any more precisely without significantly increasing the complexity of the calculations.

IV. SUPPRESSION OF THE ZIGZAG INSTABILITY

Consider a zigzag phase disturbance of the form

$$\phi^x\equiv\phi^x(Y,T)\ ,\quad \phi^y\equiv\phi^y(X,T)\ . \tag{46}$$

It can be shown that ϕ^x grows at a rate $\sigma_1|_{k=0}=-Kl^2$, and ϕ^y grows at a rate $\sigma_1|_{l=0}=-Kk^2$. If a traditional zigzag instability were to occur, we would expect the growth rate of a zigzag disturbance to be zero at this order, since there are no spatial derivatives of higher order than second in Eqs. (2)–(5). A traditional zigzag instability depends on the presence of third- and fourth-order spatial derivatives in the amplitude equations, and occurs only for patterns at wavelengths longer than critical ($q<0$). In this case, the growth rates are nonzero because of the presence of the term $-K(k^2+l^2)$ in σ_1, which comes from the imaginary part of the transverse diffusion, i.e., the term $i\alpha A_{YY}$ in Eq. (2), and the corresponding terms in Eqs. (3)–(5). In addition, the pattern is stable to zigzag-type disturbances if $K\equiv\alpha\lambda_1>0$, and unstable if $K<0$, for all wavelengths. Clearly, this is not a traditional zigzag instability. In fact, the zigzag-type phase disturbance turns out to be a special case of the rectangular phase mode, described below, which also grows at a rate σ_1. In other words, the effect of the imaginary transverse diffusion suppresses the zigzag instability.

V. RECTANGULAR INSTABILITY

There is a bifurcation at

$$2(M+N-K)k^2l^2=-K(k^2+l^2)^2\ , \tag{47}$$

corresponding to $\sigma_1=0$. In the long-wavelength limit $k,l\to0$, the instability has the eigenmode

$$(\hat\theta+\hat\psi+\hat\chi+\hat\phi)=0\ , \tag{48}$$

$$k(\hat\theta-\hat\psi)=-l(\hat\chi-\hat\phi)\ . \tag{49}$$

In terms of the spatial phases, this can be written

$$k\hat\phi^x=-l\hat\phi^y\ . \tag{50}$$

This mode corresponds to the rectangular Eckhaus instability of stationary squares, (see Hoyle [19]). It represents local stretching along one coordinate axis, and contraction along the orthogonal axis. The effect of the instability on a standing-square pattern is to produce rhombi.

The relationship between the two spatial phases can be rewritten

$$\phi^x_X = -\phi^y_Y \ . \tag{51}$$

It is now easy to see that a zigzag phase disturbance, which satisfies

$$\phi^x_X = -\phi^y_Y = 0 \ , \tag{52}$$

is a special case of a rectangular disturbance.

The rectangular instability boundary is parabolic in (r,q) space, and can be written

$$K(1+\gamma^2)^2 + 2(M+N-K)\gamma^2 = 0 \ , \tag{53}$$

for $k=\gamma l$. This shows that the pattern becomes unstable to modes with different k/l ratios at different points, as illustrated in Fig. 1.

Considering squares at the critical wave number $q=0$, and supercritical values of the bifurcation parameter $r>0$, we find that if $\alpha\lambda_1<0$, then the pattern is unstable to the rectangular mode for all $r>0$. If $\alpha\lambda_1>0$ and $f_3+2\lambda_1\eta_i>0$, then the pattern is stable to this mode for all $r>0$, but if $\alpha\lambda_1>0$ and $f_3+2\lambda_1\eta_i<0$, then the pattern is unstable to the rectangular mode for $0<r<r_1$, and stable for $r>r_1$, where

$$r_1 \equiv \frac{c^2 f_2(f_3+2\lambda_1\eta_i)}{4f_3(\eta_r+\lambda_3\eta_i)(1+2\alpha\lambda_1)} \ . \tag{54}$$

VI. OSCILLATORY INSTABILITY

The presence of a codimension-2 bifurcation is indicated by the forms of σ_2 and σ_3. There is a stationary bifurcation at

$$AH(k^2+l^2) = -\{(G+M+HW+AZ)(k^2+l^2)/2 - (M+N-K)k^2l^2/(k^2+l^2)\}^2 \ , \tag{55}$$

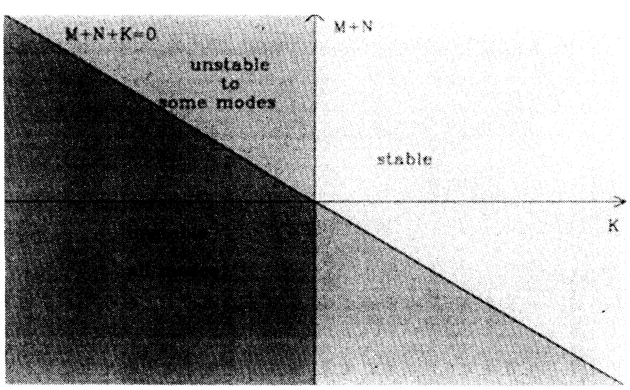

FIG. 1. In the rectangular instability of standing squares, the pattern becomes unstable to different shapes of perturbation, as characterized by the ratio k/l, in different regions of parameter space.

and a Hopf bifurcation at

$$(G+M+HW+AZ)(k^2+l^2)^2 - 2(M+N-K)k^2l^2 = 0 \ , \tag{56}$$

where $AH>0$. The relationship between the two instabilities is illustrated in Fig. 2. We shall be interested primarily in the Hopf bifurcation, since this occurs for $AH=O(1)$, which is the general case.

In the long-wavelength limit, the eigenmodes corresponding to σ_2 and σ_3 satisfy

$$(\hat\theta+\hat\psi+\hat\chi+\hat\phi) = \pm(H/A)^{1/2}(k^2+l^2)^{-1/2} \times[k(\hat\theta-\hat\psi)+l(\hat\chi-\hat\phi)] \ , \tag{57}$$

$$l(\hat\theta-\hat\psi) = k(\hat\chi-\hat\phi) \ . \tag{58}$$

Writing this shape in terms of the spatial and temporal phases, we find that

$$\sigma\hat\phi^x = iAk\hat\phi^t \ , \tag{59}$$

$$\sigma\hat\phi^y = iAl\hat\phi^t \ , \tag{60}$$

$$l\hat\phi^x = k\hat\phi^y \ . \tag{61}$$

Locally, this mode rotates the roll axes toward or away from each other; in a standing-square planform the effect is to create rectangles. The temporal phase is also involved, and at the Hopf bifurcation this leads to a quasiperiodic oscillation of the bifurcated state.

Note that the relationship between the two spatial phases (61) can be rewritten $\phi^x_Y=\phi^y_X$. So we see that the two-dimensional oscillatory Eckhaus instability, with $\phi^x\equiv\phi^x(X,T)$ and $\phi^y\equiv\phi^y(Y,T)$, is a special case of this three-dimensional oscillatory instability.

When $AH<0$, one of σ_2 and σ_3 is positive and the other negative; however, when $AH>0$, σ_2 and σ_3 are complex and the Hopf bifurcation may occur. In the limit of large c, where the physical dimensional phase velocities are $O(1)$, it is easily seen that $AH\to c^2/2$, so the Hopf bifurcation is expected to occur.

For $AH>0$, the second-order real part of σ_2 and σ_3 passes through zero when

$$(G+M+HW+AZ)(1+\gamma^2)^2 - 2(M+N-K)\gamma^2 = 0 \ , \tag{62}$$

for $k=\gamma l$. Once again the pattern becomes unstable to

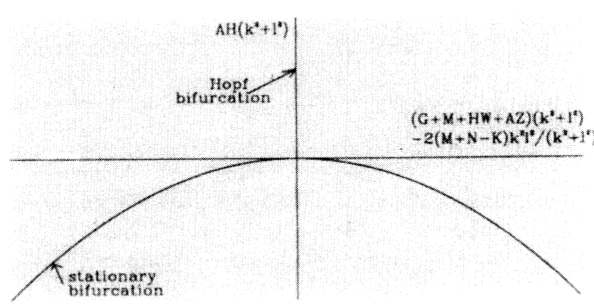

FIG. 2. Codimension-2 bifurcation for the phase modes of standing squares.

modes with different k/l at different points, as shown in Fig. 3.

For squares at the critical wave number, $q=0$, and $r>0$, we find that the system is unstable to the oscillatory mode if $1+\alpha(\lambda_1+\lambda_2)<0$. If $1+\alpha(\lambda_1+\lambda_2)>0$, and $f_3+2\lambda_1\eta_i>0$, then the pattern is stable to the oscillatory mode. However, if $1+\alpha(\lambda_1+\lambda_2)>0$ and $f_3+2\lambda_1\eta_i<0$, then we find that standing squares are unstable to the oscillatory mode for $0<r<r_2$, and stable for $r>r_2$, where

$$r_2 \equiv \frac{c^2 f_2(f_3+2\lambda_1\eta_i)}{4f_3(\eta_r+\eta_i\lambda_3)[3+2\alpha(\lambda_1+\lambda_2)]} . \tag{63}$$

It is interesting to compare these stability criteria with those found by Coullet, Fauve, and Tirapegui [15] for a single standing wave. They found that the standing wave is unstable to long-wavelength, low-frequency modulations if

$$1+\alpha\left[\frac{\nu_r\nu_i-\mu_r\mu_i}{\nu_r^2-\mu_r^2}\right]<0 , \tag{64}$$

using the notation of this paper. In the standing-square system, this would correspond to

$$1+\alpha(\lambda_1+\lambda_2)/2<0 , \tag{65}$$

where we have set $\gamma=\eta=0$, so that there is no coupling between the two orthogonal standing waves. Note that when $\eta=0$, the amplitude stability condition $f_3-2\eta_r>0$ reduces to $f_3>0$, and in turn the condition for instability to the oscillatory mode reduces to

$$1+\alpha(\lambda_1+\lambda_2)<0 . \tag{66}$$

Comparing this with the condition (65), we see that the condition for instability of standing squares to the oscillatory mode is less stringent than that for instability of a single standing wave to this long-wavelength oscillatory mode. The difference between conditions (65) and (66) is a result of the phase locking, $(\theta+\psi-\chi-\phi)\ll 1$, which must therefore be destabilizing.

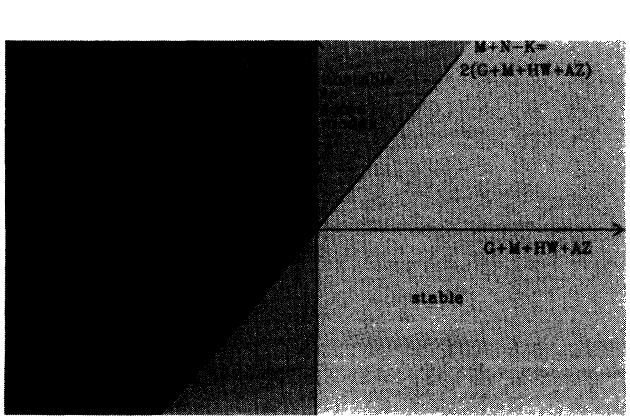

FIG. 3. In the oscillatory phase instability of standing squares, the pattern becomes unstable to different shapes of perturbation, as characterized by the ratio k/l, in different regions of parameter space.

VII. GEOMETRICAL INTERPRETATION OF THE PHASE-INSTABILITY EIGENMODES

The spatial structure of the phase-instability eigenmodes can be interpreted geometrically. We can interpret the rectangular phase mode as a divergence-free disturbance, and the oscillatory mode as a curl-free disturbance. Writing

$$\boldsymbol{\phi}=\begin{bmatrix}\phi^x\\ \phi^y\end{bmatrix} , \tag{67}$$

we find that for the rectangular instability we have

$$\nabla\cdot\boldsymbol{\phi}=0 , \tag{68}$$

and for the oscillatory instability

$$\nabla\times\boldsymbol{\phi}=\mathbf{0} . \tag{69}$$

A simple curl-free dístrubance is given by $\boldsymbol{\phi}=\mathbf{X}$, where $\mathbf{X}=(X,Y)$; this is a radial phase vector. Similarly, a simple divergence-free disturbance is given by $\boldsymbol{\phi}=\hat{\mathbf{z}}\times\mathbf{X}$, where $\hat{\mathbf{z}}$ is a unit vector in the direction orthogonal to the XY plane; this is an azimuthal phase vector. A structure with a spiral phase vector can be produced by combining divergence-free and curl-free disturbances, such as the two above.

These patterns are particularly interesting in connection with the phase waves seen by Nasuno, Sano, and Sawada [3]. They observed long-wavelength "target" patterns (which are concentric ellipses propagating radially outwards), and rotating spiral patterns, superimposed upon a standing rectangular pattern. In addition to being long wavelength with respect to the original pattern, they also have a lower frequency of oscillation. I suggest that these patterns might be connected with curl-free phase disturbances (for target patterns) and a combination of divergence-free and curl-free phase disturbances (for spirals). Their lower frequency of oscillation would correspond to the low-frequency modulation which leads to quasiperiodic oscillations at the oscillatory instability.

VIII. DISCUSSION

A standing-square or alternating-roll pattern can arise, in certain parameter regimes, as the stable preferred planform at a Hopf bifurcation from the conductive solution. Its subsequent evolution can be described by the amplitude Eqs. (2)–(5).

Standing squares and alternating rolls exhibit the same three-dimensional rectangular Eckhaus instability as a stationary square pattern. Locally, this mode produces stretching along one coordinate axis, and contraction along the other. There is a new oscillatory instability which is locally equivalent to rotating the coordinate axes toward or away from each other. The oscillatory instability also involves a change in the frequency of oscillation as the instability progresses; in other words, it results in quasiperiodic oscillations. This is similar to the result found by Coullet, Fauve, and Tirapegui [15] for the two-dimensional case. However, the criterion for instability of the two-dimensional standing wave is more stringent than the criteria for instability of standing squares and alternating rolls to the rhombic oscillatory mode; this is the destabilizing effect of the phase locking, $(\theta+\psi-\chi-\phi)\ll 1$.

The rectangular mode can be interpreted as a divergence-free phase disturbance, and the oscillatory mode as a curl-free disturbance. Structures with spiral phase vectors can be formed from a combination of the oscillatory and rectangular modes. It is possible that these spiral structures are connected with the spiral phase waves seen by Nasuno, Sano, and Sawada [3] in their experiments on electrohydrodynamic convection of nematic liquid crystals, and that the curl-free disturbances are relevant to the target patterns they observe. These phase waves are long wavelength with respect to the underlying standing rectangular pattern, and also have a lower frequency of oscillation, just like the phase disturbances described in this paper.

The rectangular and oscillatory instability boundaries can interact in numerous different ways, depending on the parameter values. We have only considered in detail the case of a pattern at the critical wave number, yet even here the number of parameter regimes with qualitatively different stability diagrams is very large.

We also find that the zigzag instability is suppressed by the effect of imaginary transverse diffusion in the amplitude equations, and the two-dimensional Eckhaus instability is subsumed into the three-dimensional oscillatory instability.

Numerical simulations of the phase instabilities are currently underway, and it is hoped to present these in a future paper.

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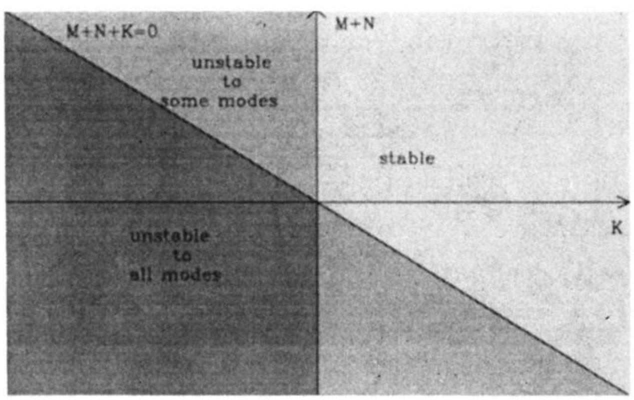

FIG. 1. In the rectangular instability of standing squares, the pattern becomes unstable to different shapes of perturbation, as characterized by the ratio k/l, in different regions of parameter space.

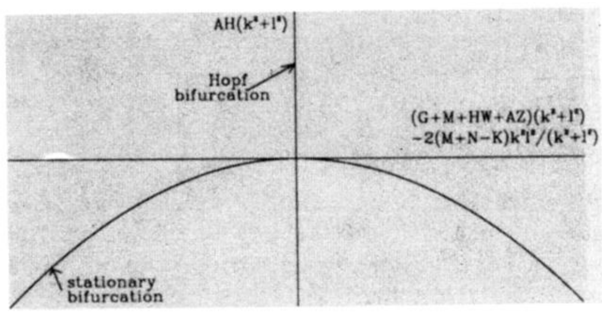

FIG. 2. Codimension-2 bifurcation for the phase modes of standing squares.

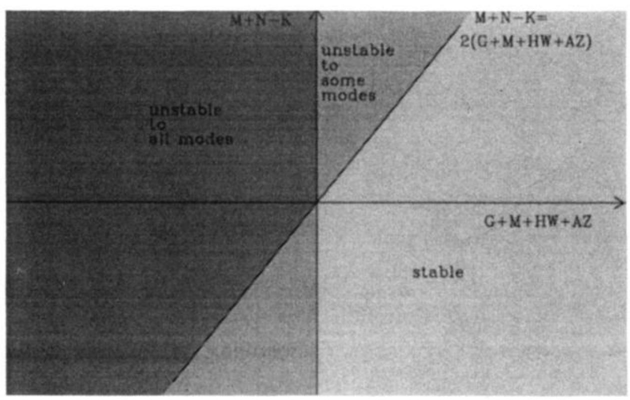

FIG. 3. In the oscillatory phase instability of standing squares, the pattern becomes unstable to different shapes of perturbation, as characterized by the ratio k/l, in different regions of parameter space.